→ Conservation and Symmetry { Read LL Chapt. 2

Point: i) Conservation laws (familiar and otherwise) emerge/result from symmetries.

ii) Symmetry → invariance L w/r q → conserved momentum/quantity

⇒ Noether's Thm.

① → Energy (homogeneity of time, for closed system)

- $\partial_t L = 0$ ⇔ homogeneity of time unless broken by external input

↓

- energy conservation

$$\frac{dL}{dt} = \cancel{\frac{\partial L}{\partial t}} + \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q}$$

0

can shift $t=0$, arbitrarily.

$$= \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q}$$

$\to \quad \delta L = 0 \quad$ upon $\quad t \to t + \delta t$

$$\delta L = L(t + \delta t) - L(t)$$

$$= \delta t \, \frac{\partial L}{\partial t}$$

$\therefore$ time translation invariance
$\Rightarrow$

$$\partial L / \partial t = 0$$

but

$$\frac{dL}{dt} = \frac{d}{dt} L(t, q, \dot{q})$$

$$= \cancel{\partial_t L} + \frac{\partial L}{\partial q} \dot{q} + \frac{\partial L}{\partial \dot{q}} \ddot{q}$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial q} \dot{q} + \frac{\partial L}{\partial \dot{q}} \ddot{q} - L \right) = 0$$

etc.

but $\partial L/\partial \xi = d/dt\left(\partial L/\partial \dot{\xi}\right)$

$$dL/dt = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}}\right)\dot{\xi} + \frac{\partial L}{\partial \dot{\xi}}\ddot{\xi}$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}}\dot{\xi}\right)$$

$\therefore$

$$\frac{d}{dt}\left(\dot{\xi}\frac{\partial L}{\partial \dot{\xi}} - L\right) = 0$$

conserved quantity!

So

$$\boxed{E \equiv \text{energy} = \dot{\xi}\frac{\partial L}{\partial \dot{\xi}} - L}$$

conserved for $\partial_t L = 0$

defines conservative system.

Note: Will see:

- Hamiltonian H defined s/t
  $H = p\dot{\xi} - L$
  where $p \equiv \partial L/\partial \dot{\xi}$ and eliminate $\dot{\xi}$ in favor p

- in general

$H \neq E$ ($H = E$ for conservative)

- so, <u>can</u> define H when $\partial_t L \neq 0$

and energy not conserved.

Hamiltonian $\not\Rightarrow$ energy conservation

(b) - <u>Linear Momentum</u>

⇒ For closed system, <u>homogeneity of space</u> ⇒ mechanical properties unchanged by displacement (i.e. parallel displacement) of system in space

i.e. can shift origin of coordinate system, physics invariant upon
$\underline{r} \to \underline{r} + \underline{\epsilon}$

so

$$\delta L = \sum_i \frac{\partial L}{\partial \underline{r}_i} \cdot \underline{\epsilon} = \underline{\epsilon} \cdot \sum_i \frac{\partial L}{\partial \underline{r}_i}$$

→ Translation

L invariant under $\underline{r} \to \underline{r} + \delta\underline{r}$

$$\delta L = L(\underline{r} + \delta\underline{r}) - L(\underline{r})$$

$$= \frac{\partial L}{\partial \underline{r}} \cdot \delta\underline{r}$$

but $\delta L = 0 \iff \partial L / \partial \underline{r} = 0$

and $\frac{d}{dt}\left(\partial L / \partial \underline{v}\right) = \frac{\partial L}{\partial \underline{r}} = 0$

etc.

$$\delta L = 0 \Rightarrow \sum_i \frac{\partial L}{\partial \underline{r}_i} = 0$$

$$\text{so L.E.O.M.} \Rightarrow \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \underline{v}_i}\right) = 0$$

$$\partial L / \partial \underline{v}_i = \underline{p}_i \rightarrow \text{generalized momentum}$$

so

$$\boxed{\frac{d}{dt}\underline{P} = 0 \qquad \underline{P} = \sum_i \partial L / \partial \underline{v}_i}$$

Momentum conserved ⟸ follows from spatial homogeneity

and for:

$$\underline{F}_i = -\partial U / \partial \underline{r}_i = \partial L / \partial \underline{r}_i = 0$$

⇕

gen. force = 0 ⟺ homog. space.

generalized momentum conserved.

and thus:

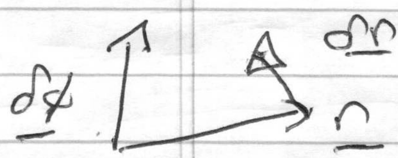
c.) – Angular Momentum

⇒ isotropy of space, for closed system

∴ physics invariant under infinitesimal rotation

i.e. $\delta\underline{\phi} \equiv$ vector infinitesimal rotation

$|\delta\phi| \rightarrow$ magnitude, $\dfrac{\delta\underline{\phi}}{|\delta\phi|} \rightarrow$ direction of axis



$$\delta\underline{r} = \delta\underline{\phi} \times \underline{r}$$

$$\delta\underline{v} = \delta\underline{\phi} \times \underline{v}$$

so, isotropy ⇒ [redacted] L invariant, $\delta L = 0$ upon infinitesimal rotation

⇒

$$\delta L \Big|_{\text{rotation}} = \sum_i \left(\frac{\partial L}{\partial \underline{r}_i} \cdot \delta \underline{r}_i + \frac{\partial L}{\partial \underline{v}_i} \cdot \delta \underline{v}_i \right) = 0$$

$$= 0 = \sum_i \left(\underline{\dot{p}}_i \cdot \delta\underline{\phi} \times \underline{r}_i + \underline{p}_i \cdot \delta\underline{\phi} \times \underline{v}_i \right)$$

re-arranging:

$$\delta H_{rot} = 0 = \delta\underline{\phi} \cdot \left(\sum_i \underline{r}_i \times \underline{\dot{p}}_i + \underline{\dot{r}}_i \times \underline{p}_i \right)$$

$$= \delta\underline{\phi} \cdot \frac{d}{dt} \sum_i (\underline{r}_i \times \underline{p}_i)$$

$$\boxed{\begin{gathered} \delta H_{rot} = 0 \Rightarrow \frac{d}{dt}\underline{L} = 0 \\ \underline{L} = \sum_i \underline{r}_i \times \underline{p}_i \end{gathered}}$$

$\therefore$ rotational invariance $\Rightarrow$ angular momentum conserved.

Note: Angular momentum depends on choice of origin, except when system at rest, as a whole.

i.e. $\underline{L} = \sum_i \underline{r}_i \times \underline{p}_i$

$$\underline{r} \rightarrow \underline{r}' + \underline{a}$$

so

$$\underline{L} = \sum_i \underline{r}_i' \times \underline{p}_i + \sum \underline{a} \times \underline{p}_i$$

$$= \underline{L}' + \underline{a} \times \underline{P} \rightarrow \text{origin dep.}$$

<u>So observe</u>

- time homogeneity $\Rightarrow$ $\underset{t \to t+\delta t}{\delta L = 0} \rightarrow \partial_t L = 0$

  $\Rightarrow$ energy conservation

- spatial homogeneity $\rightarrow$ $\underset{\underline{x} \to \underline{x}+\delta \underline{x}}{\delta L = 0} \rightarrow$

  $\partial_{\underline{x}} L = 0$ $\Rightarrow$ linear momentum conservation

- rotational isotropy $\rightarrow$ $\underset{\underline{\phi} = \underline{\phi}+\delta\underline{\phi}}{\delta L = 0} \rightarrow$

  $\partial L / \partial \phi = 0$ $\Rightarrow$ angular momentum conserved.

$\Rightarrow$ Suggest Connection:

symmetry $\rightarrow$ ignorable or 'cyclic' coordinate ($\partial L / \partial q = 0$) $\Rightarrow$ conservation law

$\boxed{\Rightarrow \text{Noether's Theorem}}$

Precise statement:

If the functional:

$$S = \int_a^b L(t, q^\mu, \dot{q}^\mu)\,dt$$

is an extremal and if under the infinitesimal transformation:

$$t' = t + \epsilon\tau + \ldots$$

$$q^{\mu\prime} = q^\mu + \epsilon\rho^\mu + \ldots$$

the functional is invariant according to the definition (allowing for the inhomogeneous case) (total time deriv.)

$$L'\,dt'/dt - L = \epsilon\,dF/dt + O(\epsilon^s)$$

$s > 1$

then:

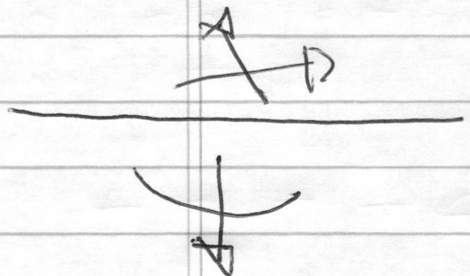
$$p_\mu \rho^\mu - H\tau - F = \text{const}$$

holds and constitutes a conservation law.

→ Symmetry Exercises (L+L / Problem 3, pg. 21)

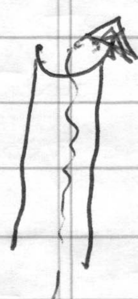
? which components of $\underline{P}$, $\underline{L}$ conserved in following fields.
(think as $U = \text{const}$ zones)

a.) infinite homogeneous plate



- 2 components $\underline{P}$ in plane // plate
- $\underline{L}_{\perp}$ component $\perp$ to plate plane

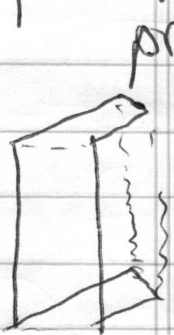
b.)



infinite homogeneous cylinder

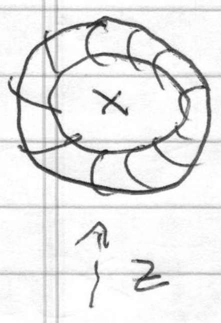
- $\underline{P}$ component along axis of cylinder
- $\underline{L}$ component along axis of cylinder

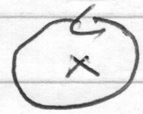
c.) infinite homogeneous rectangular prism



– p_z parallel to prism.

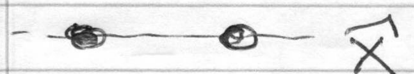
d.) Torus L_z, only





– L_z only (i.e. toroidal angular momentum)

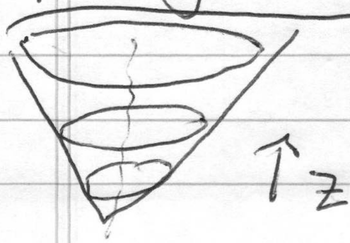
e.) Two points



– L_x

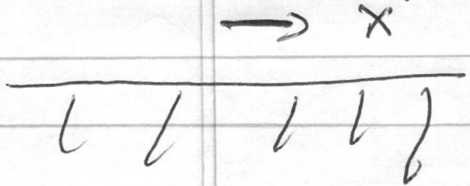
(angular momentum for rotations about axis of line joining 2 pts.).

f.) homogeneous cone



– L_z, only

g.) infinite half plate (2D)



— P_x, only

h.) infinite homogeneous cylindrical helix, pitch h

$\hat{z}$

$h \equiv$ pitch (vertical distance in 1 rotation)

Invariance:

$\delta\phi$ rotation $+ \frac{\delta\phi}{2\pi} h$ translation

i.e. $$\delta L = \frac{\partial L}{\partial z}\delta z + \frac{\partial L}{\partial \phi}\delta\phi$$

$$dz = \frac{h}{2\pi}\delta\phi$$

$$\partial L/\partial z = d/dt(p_z), \quad \partial L/\partial\phi = \frac{d}{dt}(L_z)$$

so $$\delta L = \dot{p}_z\left(\frac{h}{2\pi}\delta\phi\right) + \dot{L}_z\,\delta\phi$$

$$= \delta\phi\left(h/2\pi\,\dot{p}_z + \dot{L}_z\right)$$

so $\dot{L} = 0 \Rightarrow \left(\frac{h}{2\pi} p_z + L_z \right) = 0$

$\frac{h}{2\pi} p_z + L_z = \text{const.}$

→ Scale Symmetry and Virial Theorems

- scale symmetry of interest,
  as well space-time symmetry.

- scale symmetry ⟺ how exploit
  dimensional analysis in framework
  of Lagrangian mechanics.

- scale symmetry ⟺ related to
  virial theorems → useful in
  Astrophysics.

Scale Symmetry / Virial Theorems

→ where dimensional analysis
  meets symmetry constraints.

Re: scale symmetry ⇒

if $U(\alpha\underline{r}_1, \alpha\underline{r}_2, \ldots \alpha\underline{r}_n) = \alpha^k U(\underline{r}_1, \underline{r}_2, \ldots \underline{r}_n)$

i.e. re-scale variables

⇒ U is $\boxed{\text{homogeneous.}}$ (related to scale invariance)

Many relevant U are homogenous, i.e.

harmonic oscillator: $k=2$

Coulomb / gravity: $k=-1$

rescale variables ⇒ same U!

etc. ⇒ homogenous U ⇒ power law structure.

Now, recall:

EOM from $\delta \int dt\, L = 0$, so if

$L \to \sigma L$, EOM unchanged!

σ: const. factor

⇒ Multiplying Lagrangian by constant factor leaves physics unchanged.

→ so, for homogeneous U, can generate class of rescalings which multiply Lagrangian by const. factor

⇒ same EOM!

→ $\boxed{\text{such rescalings define basic class of relations between quantities.}}$

⇒ useful for basic story/characteristics w/o detailed crank. → insight!

Now: $$S = \int dt \left(\frac{1}{2} m \dot{\underline{r}}^2 - U(\underline{r}) \right)$$

$r \to \alpha r'$

$t \to \beta t'$

$$S = \int dt \left(\frac{1}{2} m \frac{\alpha^2}{\beta^2} \dot{r}'^2 - \alpha^k U(r') \right)$$

so if $\alpha^2/\beta^2 \sim \alpha^k \Rightarrow L$ multiplied by net factor and δS invariant.

$$\Rightarrow \boxed{\beta \sim \alpha^{1-k/2} \text{ defines time-space rescaling leaving EOM unchanged.}}$$

i.e. $\boxed{t'/t \sim (\ell'/\ell)^{1-k/2}}$

→ works only for homog. fctn.

Equivalently:

$v'/v \sim \alpha^{k/2} \sim (\ell'/\ell)^{k/2}$
(velocity)

$E'/E \sim (\ell'/\ell)^{k}$

$L'/L \sim (\ell'/\ell)^{1+k/2}$
(angular momentum)

→ Eg. ⓐ $U \sim z$ gravity $k \sim 1$

$\Rightarrow t'/t \sim (\ell'/\ell)^{1/2}$

fall time ~ square of amplitude.

ⓑ $U \sim r^2$ $\qquad k = 2$ $\qquad$ (h.o.)

$t'/t \sim \ell^0$ $\Rightarrow$ period indep. of amplitude.

ⓒ $U \sim r^{-1}$ $\qquad k = -1$

$$t'/t \sim (\ell'/\ell)^{3/2}$$

Period $\sim$ (radius)$^{3/2}$ $\Rightarrow$ Kepler's 3rd Law.

$\boxed{\text{Homogeneous } U \Rightarrow \text{Virial Thms}}$!

What is a Virial Thm ?

- consider a system of particles

~ Action

$$-\ \frac{d}{dt}\Big(\sum_i \underline{p}_i \cdot \underline{x}_i\Big) = \sum_i \big(\underline{p}_i \cdot \dot{\underline{x}}_i\big) + \sum_i \big(\dot{\underline{p}}_i \cdot \underline{x}_i\big)$$

$$= \underbrace{2T}_{KE} - \sum_i \frac{\partial U}{\partial \underline{x}_i} \cdot \underline{x}_i$$

Now consider $\langle \left(\sum_i \dot{\underline{p}_i \cdot \underline{x}_i} \right) \rangle$ → time avg. of deriv.

$$\langle A \rangle = \frac{1}{\tau} \int_0^\tau A \quad \text{as } \tau \to \infty$$

So if $\sum_i \underline{p}_i \cdot \underline{x}_i$ bounded in time

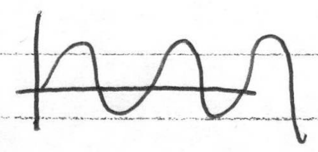
$$\left\langle \frac{d}{dt} \sum_i \underline{p}_i \cdot x_i \right\rangle = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau dt \left(\frac{d}{dt} \sum_i \underline{p}_i \cdot \underline{x}_i \right)$$

$$= \sum_i \underline{p}_i \cdot \underline{x}_i \Big|_0^\tau \to 0$$

bnded, by hypoth

$$1/\tau \left(\sum_i \underline{p}_i \cdot x_i \Big|_0^\tau \right) \to 0$$

so



$$2\langle T \rangle = \left\langle \sum_i \frac{\partial U}{\partial \underline{x}_i} \cdot \underline{x}_i \right\rangle$$

Now, if $U = U(\underline{x}_1, \underline{x}_2, \ldots x_n)$
and U homogeneous:

i.e. $U(\alpha \underline{x}_1, \ldots \alpha x_n) = \alpha^k U(\underline{x}_1, \ldots x_n)$

then $\boxed{2\langle T \rangle = k \langle U \rangle}$.

and also, of course, have:

$$T + U = E = \langle T \rangle + \langle U \rangle$$

so

$$2\langle T \rangle = k\langle U \rangle$$

$$E = \langle T \rangle + \langle U \rangle$$

$$\Rightarrow \boxed{\begin{array}{l} \langle U \rangle = \dfrac{2}{k+2} \text{ [redacted] } E \\[2ex] \langle T \rangle = (k) E / k+2 \end{array}}$$

check:

$\rightarrow$ $k = 2$ $\qquad \langle U \rangle = E/2$

$\qquad\qquad\qquad \langle T \rangle = E/2$

equipartition in H.O.

$\rightarrow$ $k = -1$

$$\langle U \rangle = E$$

$$\langle T \rangle = -E$$

$\rightsquigarrow \langle T \rangle = -E$ $\Rightarrow$ total energy negative for gravitationally bound cluster

i.e. must have boundstate for time avg. to converge.

Who cares / why care?

- virial thms relate energies to potential structure

- can relate measured K.E. (Doppler spectroscopy) to energies

- virial is single # characterizing a cluster

$$f(\underline{x}, \underline{v}, t) \rightarrow V(\underline{x}, t) \rightarrow \langle T \rangle, \langle U \rangle, E$$

Boltzmann $\rightarrow$ fluid $\rightarrow$ virial

$\downarrow$ velocity moment

$\downarrow$ $\sum_i \rightarrow$ integral in space,